Fermion masses and mixings

- Cheng-Li 11.3
- Schwartz 29

For now, we have interacting fermions, but they are massless. The fact that the left & right fermions transform under different representations of the SM group forbids any mass term.

Moreover, there is a large $U(3)^5$ global symm. given by rotating the 3 copies of each fermion q_L, u_R, d_R, l_L, e_R.

But there are missing operators in the Lagrangian, involving two fermions and a Higgs.
By power counting, these type of ops (Yukawa op) are the only missing term with dim ≤ 4.
Due to $SU(3)_c$, there are no Yukawas with

quark & lepton.

- Reminder: the bilinear $\bar{\psi}\psi$, in terms of L & R fields is $\bar{\psi}\psi = \bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L$, with

$$\psi_{L,R} = P_{L,R}\,\psi \quad ; \quad P_{L,R} = \frac{1 \mp \gamma^5}{2}$$

For $\bar{\psi}$,

$$\bar{\psi}_L = (\psi_L)^\dagger \gamma^0 = \psi^\dagger P_L \gamma^0 = \psi^\dagger \gamma^0 P_R = \bar{\psi} P_R$$

and similar for ψ_R,

$$\bar{\psi}_R = \bar{\psi} P_L$$

Therefore, expanding $\bar{\psi}\psi$ only cross terms survive.

A term like

$$\mathcal{L} \supset -m\bar{\psi}\psi$$

or

$$\mathcal{L} \supset -m\bar{\psi}_L \psi_R + h.c.$$

is called a Dirac mass term.

It requires <u>both</u> chiralities

- There is a second type of mass term for fermions. Recall that the conjugate

$$\psi^c = C\bar{\psi}^T$$

where $C = i\gamma^2\gamma^0$ is the charge conjugation matrix. Note that terms like

$$\mathcal{L} \supset -m\bar{\psi}\psi^c$$

are allowed by Lorentz invariance. These are called Majorana mass terms. They break "fermion number" by 2 units.
If ψ is charged under a gauge group, Majorana mass terms are forbidden.
Therefore, in the SM, there are no Majorana mass terms.
We will eventually discuss Majorana mass terms and ν_R.

- Lepton Yukawas.

We must form an $SU(2)\times U(1)$ invariant from

$$\ell_L \in 2_{-1/2}\ ,\quad e_R \in 1_{-1}\ ,\quad H \in 2_{1/2}$$

Due to Lorentz,

$$\bar{e}_R\, \ell_{L,\alpha}$$
α: SU(2) index

$\bar{\ell}_L^{\alpha} e_R$ is in the conjugate.

$\bar{e}_R e_R^c$ and $\bar{\ell}_L \ell_L^c$ cannot form a singlet with only H.

We can write

$$\mathrm{I}: \quad (H^*)^{\alpha}\, \bar{e}_R\, \ell_{L\alpha} \quad \longrightarrow \text{hyp } -\tfrac{1}{2}+1-\tfrac{1}{2}=0$$

$$\mathrm{II}: \quad H_{\alpha}\, \epsilon^{\alpha\beta}\, \bar{e}_R\, \ell_{L\beta} \quad \longrightarrow \text{hyp } \tfrac{1}{2}+1-\tfrac{1}{2}\neq 0$$

So we have a unique operator in the lepton sector, given by the yukawa

$$\mathcal{L} \supset -y_e \bar{e}_R H^* l_L - y_e^* \bar{l}_L H e_R$$

- In the unitary gauge,

$$H = \begin{pmatrix} 0 \\ \frac{v+h}{\sqrt{2}} \end{pmatrix} , \quad l_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$$

$$\mathcal{L} \supset -y_e \frac{v+h}{\sqrt{2}} \bar{e}_R e_L - y_e^* \frac{v+h}{\sqrt{2}} \bar{e}_L e_R$$

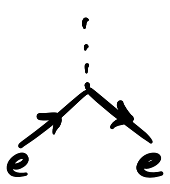
$$= -\frac{y_e v}{\sqrt{2}} \bar{e} e - \frac{y_e h}{\sqrt{2}} \bar{e} e$$

where in the last line we used that y_e can be made real by redefining the phase of, say, e_R.

So the charged lepton gets a mass

$$m_e = \frac{y_e v}{\sqrt{2}}$$

and a coupling to the Higgs given by



$$= -i \frac{y_e}{\sqrt{2}} = -i \frac{m_e}{v}$$

The neutral lepton, the ν, does not get a mass.

- Experimentally, neutrinos have been observed to have a non-zero mass. So the above seems to lead to wrong conclusions. The neutrino masses are much smaller than the masses of the charged fermions, so we will just say that at "leading order" the masslessness of ν's is correct.
  Leading order in what, you might ask. Time permits, we might go back to this point.

- The masses of the charged leptons is an input of the SM, it cannot be predicted within its framework.

They are observed to be

$$y_e = \frac{\sqrt{2}\, m_e}{v} = (\underset{e}{3 \cdot 10^{-6}} , \underset{\mu}{6 \cdot 10^{-4}} , \underset{\tau}{10^{-2}})$$

We do not understand the origin of these numbers, neither why are small.

- Quark Yukawa

In this case we have

$$q_L \in 2_{1/6} \quad , \quad u_R \in 1_{2/3} \quad , \quad d_R \in 1_{-1/3}$$

There are two possible structures, the "down type" and the "up type" yukawas.

$$\mathcal{L}_y^{(d)} = - y_d \, \bar{q}_L^{\alpha} H_\alpha d_R + h.c. \qquad \left(-\tfrac{1}{6} + \tfrac{1}{2} - \tfrac{1}{3} = 0 \right)$$

$$\mathcal{L}_y^{(u)} = - y_u \, \bar{q}_L^{\alpha} H_\alpha^{c} u_R + h.c. \qquad \left(-\tfrac{1}{6} - \tfrac{1}{2} + \tfrac{2}{3} = 0 \right)$$

where we introduced the notation

$$H^c_\alpha \equiv \epsilon_{\alpha\beta}(H^*)^\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} h_u^* \\ h_d^* \end{pmatrix}$$

so H^c is a doublet like H but with hypercharge $-1/2$.

You can remember that

$$\sigma_2 \tau^{a*} \sigma_2 = -\tau^a \rightarrow \sigma_2 e^{-i\alpha_a \tau^{a*}} \sigma_2 = e^{+i\alpha_a \tau^a}$$

so

$$H^c = i\sigma_2 H^* \longrightarrow i\sigma_2 e^{-i\alpha_a \tau^{a*}} H^* = e^{i\alpha_a \tau^a} i\sigma_2 H^*$$
$$= e^{i\alpha_a \tau^a} H^c$$

- In the unitary gauge,

$$H^c = \begin{pmatrix} \frac{v+h}{\sqrt{2}} \\ 0 \end{pmatrix}$$

so that $\mathcal{L}_y^{(u)}$ gives a mass for the up component of the fermion doublet, while $\mathcal{L}_y^{(d)}$ gives mass to the down component, in the same way as for leptons.

$$\mathcal{L}_y^{(d)} = \frac{-y_d v}{\sqrt{2}} \bar{d} d - \frac{y_d h}{\sqrt{2}} \bar{d} d$$

$$\longrightarrow \quad m_{u,d} = y_{u,d} \frac{v}{\sqrt{2}}$$

$$\mathcal{L}_y^{(u)} = -\frac{y_u v}{\sqrt{2}} \bar{u} u - \frac{y_u h}{\sqrt{2}} \bar{u} u$$

The Yukawas are small for all quarks, except for the top quark, for which

$$m_t \simeq 173\,\text{GeV} \longrightarrow y_t \simeq 1.$$

The top yukawa is the strongest coupling in the SM.

For the up & down quarks, $y \sim 10^{-5}$. Like for the leptons, we do not understand the hierarchy of numbers.

- Family structure, lepton sector.

The SM has 3 copies of each fermion. Therefore, the yukawa couplings are in fact 3×3 matrices.

But not all entries of the matrix are physical.

We can do field redefinitions without changing the physical observables.

Take the most general lepton Yukawa matrix

$$\mathcal{L} \supset -(Y_e)_i{}^j \, \bar{l}_L{}^i H e_{Rj} - (Y_e^\dagger)^i{}_j \, \bar{e}^j H^\dagger l_{Li}$$

with $i = 1, 2, 3$.

It seems that we have 9 complex parameters to specify.

Note that the kinetic term & gauge interactions are diagonal:

$$\mathcal{L} \supset \sum_{i=1}^{3} \bar{l}_L{}^i \, i\cancel{D} \, l_L + \sum_{j=1}^{3} \bar{e}_R{}^j \, i\cancel{D} \, e_{Rj}$$

There is a global $U(3)_L \times U(3)_e$ symmetry group.

$$l_{Li} \to (V_L)_i{}^j \, l_{Lj} \qquad l_L \in (3, 1)$$

$$\bar{l}_L{}^i \to (V_L^\dagger)^i{}_j \, \bar{l}_L{}^j \qquad l_L \in (\bar{3}, 1)$$

$$e_{R,i} \to (V_e)_i{}^j e_{Rj} \qquad e_R \in (1, 3)$$

$$\bar{e}_R{}^i \to (V_e^\dagger)^i{}_j \bar{e}_R^j \qquad \bar{e}_R \in (1, \bar{3})$$

where $V^\dagger V = 1$.

This does not change kinetic or gauge couplings, but does change the Yukawas.

However, we can always write

$$Y_e = U_L Y_e^D U_R^\dagger \qquad \text{with} \quad U_{L,R}^\dagger U_{L,R} = 1$$

by doing a singular value decomposition with

$$Y_e^D = \begin{pmatrix} y_e & 0 & 0 \\ 0 & y_\mu & 0 \\ 0 & 0 & y_\tau \end{pmatrix}$$

diagonal & with positive eigenvalues.

Then we write

$$\mathcal{L} \supset - \bar{\ell}_L H U_L Y_e^D U_R e_R + h.c.$$

we can perform a $U(3)_L \times U(3)_e$ transformation with $V_{L,e} = \mathcal{U}_{L,R}$ to get

$$\mathcal{L} \supset -\bar{l}_L H Y_e^D e_R + h.c.$$

This has important implications. Not only simplifies the Lagrangian, it tells us that the most general $d=4$ Lagrangian is diagonal in the lepton flavour space. No transitions between lepton flavors are allowed in the SM.

This is seen in the following way. There are a subset of rotations that do leave the Yukawa invariant, which is rephasing, for each eigenvector,

$$l_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \longrightarrow e^{i\alpha_l} \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$$

$$\bar{e}_R \longrightarrow \bar{e}_R e^{-i\alpha_l}$$

this is a true global symm, a $U(1)_\ell$ "lepton number".

One for each family $\Rightarrow$
- electron number
- muon number
- tau number

These are accidental symmetries of the SM, when working at dim ≤ 4. They are violated by $d > 4$ operators.

- Quark masses and mixings

The analysis of the quark sector is very different. We have

$$\mathcal{L} \supset \sum_{i=1}^{3} \bar{q}_L^i \, i\not{D} \, q_{Li} + \sum_{i=1}^{3} \bar{u}_R^i \, i\not{D} \, u_{Ri} + \sum_{i=1}^{3} \bar{d}_R^i \, i\not{D} \, d_{Ri}$$

with a flavor group $U(3)_Q \times U(3)_u \times U(3)_d$.

So we can get rid of 3 unitary matrices in the Yukawa Lagrangian.

The problem is that we would need 4 unitary matrices to diagonalize both Yukawa matrices. This arises because we have a single matrix for the left fields, so u_L & d_L rotations are identical.

By writing

$$Y_u = U_{L,u} Y_u^D U_{R,u}^\dagger \qquad Y_d = U_{L,d} Y_d^D U_{R,d}^\dagger$$

we can rotate $U_{R,u}$ and $U_{R,d}$ by a $U(3)$ rotation of u_R, d_R.

We cannot get rid of both $U_{L,u}$ & $U_{L,d}$.

Lets say we eliminate $U_{L,u}$, so that the Lagrangian is

$$\mathcal{L} \supset - \bar{q}_L H^c Y_u^D u_R - \bar{q}_L V_Q^\dagger U_{L,d} Y_d^D d_R + h.c.$$

So we end up with 6 parameters,

$$Y_U^D = \mathrm{diag}(y_u, y_c, y_t)$$

$$Y_d^D = \mathrm{diag}(y_d, y_s, y_b)$$

and a unitary matrix of new parameters, called

$$V_{CKM} \equiv V_Q^{\dagger} \cdot \mathcal{U}_{L,d} = \mathcal{U}_{L,u}^{\dagger} \, \mathcal{U}_{L,d}$$

"Cabbibo - Kobayashi - Maskawa" matrix

(if you are from Rome area, just Cabbibo matrix)

. The presence of the CKM matrix has many important implications.

One is that there is no "quark family" symmetry in the SM.

The only accidental symmetry is a global $U(1)_B$, baryon number. So the total number of quarks is conserved.

Dimension-6 operators can break this symmetry.

The second important implication of V_{CKM} is that it allows for the violation of CP.

You remember that under P,

$$\psi \xrightarrow{P} \gamma^0 \psi \quad , \quad H \xrightarrow{P} H \quad , \quad H^c \xrightarrow{P} H^c$$

so that

$$\bar\psi_1 H \left(\frac{1-\gamma^5}{2}\right) \psi_2 \xrightarrow{P} \bar\psi_1 H \frac{1+\gamma^5}{2} \psi_2$$

because $\{\gamma^0, \gamma^5\} = 0$. So Yukawa int. are not invariant under P.

Under C,

$$\psi \xrightarrow{C} C \bar\psi^T \qquad H \xrightarrow{C} H^*$$

so CP gives

$$\bar\psi_1 H \frac{1-\gamma^5}{2} \psi_2 \xrightarrow{CP} \psi_1^T H^* C \frac{1-\gamma^5}{2} C \bar\psi_2^T$$

$$= -\psi_1^T H^* \frac{1-\gamma^5}{2} \bar\psi_2^T$$

spinors anti-comm $\rightarrow$ $= +\bar\psi_2 H^* \frac{1-\gamma^5}{2} \psi_1$

$(\bar\psi_1 \gamma^5 \psi_2)^* = -\bar\psi_2 \gamma^5 \psi_1 \rightarrow$ $= \left(\bar\psi_1 H \frac{1+\gamma^5}{2} \psi_2\right)^*$

So the Yukawa operator goes to its conjugate.

The up-type Yukawa is

$$\mathcal{L}_y^{(u)} = -(Y_u^D)_{ij}\, \bar{q}_L^i H^c u_R^j \underbrace{-(Y_u^D)^*_{ij}\, \bar{u}_R^j (H^c)^* q_L^i}_{\text{h.c. of first term.}}$$

Since $Y^D = (Y^D)^*$, under CP the first term goes to the second term and vice-versa.

In the down-type, instead,

$$\mathcal{L}_y^{(d)} = -(V_{CKM}\, Y^D)_{ij}\, \bar{q}_L^i H\, d_R^j - (V_{CKM}\, Y^D)^*_{ij}\, \bar{d}_R^j H^* q_L^i$$

so the operators go to each other under CP.

If also V_{CKM} is real, then $V_{CKM} \cdot Y^D$ is real & CP is preserved.

If (and only if) $V^*_{CKM} \neq V_{CKM}$, CP is broken.

Since V_{CKM} is unitary, it has in principle many complex entries, so CP is broken.

But we did not exhaust all our field redefinitions.

The up-type Yukawa is diagonal. Similar to the lepton sector, it is invariant under a $U(1)^3$ subgroup.

Consider another $U(1)^3$ rotation for the d_R's. Their kinetic term is clearly invariant.

The down-type Yukawas break all 6 of these global symmetries down to $U(1)$ Baryon number.

These are 5 phase rotations to remove

phases in V_{CKM}.

$$\mathcal{L} \supset -\bar{q}_L H \left(V_{CKM} \cdot Y_d^D \right) d_R + h.c.$$

$$\rightarrow -\bar{q}_L H \left(e^{-i\,\mathrm{diag}(\vec{\alpha})} V_{CKM}\, e^{i\,\mathrm{diag}(\vec{\beta})} Y_d^D \right) d_R + h.c.$$

so we can transform

$$V_{CKM} \rightarrow e^{-i\,\mathrm{diag}(\vec{\alpha})} V_{CKM}\, e^{i\,\mathrm{diag}(\vec{\beta})}$$

and remove 5 phases.

- Let us count parameters.

We have an $N \times N$ unitary matrix.

- N diagonal generators are associated with phases

$$e^{i\alpha\,\mathrm{diag}(0,\dots,1,\dots,0)} = (1,\dots,e^{i\alpha},\dots,1)$$

- The rest of $N(N-1)/2$ real symmetric generators and $N(N-1)/2$ imaginary antisymm.

are associated with both a phase and an angle.

So, for an $N \times N$ unitary matrix, one has

$$\frac{N(N-1)}{2} \text{ angles}$$

$$\frac{N(N+1)}{2} \text{ phases}$$

For 3 generations, $N=3 \rightarrow$ 3 angles
6 phases.

However, we argued that we can remove 5 phases ($2N-1$ in general), so we are left with one phase

$\rightarrow$ CP violation through V_{CKM}.

- Note that with two families, one gets 1 angle + 3 phases, but we can remove all

3 phases, and one would get

$$V_{CKM} = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix}$$

where θ_c is the Cabibbo angle.

At the time Cabibbo proposed this mixing, there was no evidence for a third family, so CP is preserved.

One day, CP breaking was discovered experimentally, and Kobayashi and Maskawa counted the phases, and said it could be explained by the existence of a third family. The third family was later discovered, and K&M got the Nobel prize. Cabibbo didn't, even if K&M paper is a trivial generalization of

Cabibbo original work. People in Rome are still deeply frustrated by this.

- The physical content of V_{CKM} is then 3 angles and 1 phase. To study its phenomenological consequences, we can work out the Feynman rules.

There are two basis choices typically discussed. One is the "gauge eigenstate" basis, where the W couplings are diagonal in flavour space, up-quark masses diagonal, but down-quark Yukawas

$$\mathcal{L} \supset -\bar{q}_L H V_{CKM} Y_d^D d_R + h.c.$$

$$= \frac{v+h}{\sqrt{2}} \bar{d}_L V_{CKM} Y_d^D d_R + h.c.$$

are not diagonal, so mass terms are not diagonal and the mixing effects must be included in the propagators.

A prefered basis is the mass eigenstate basis, where we diagonalize the down-type Yukawas,

$$\begin{pmatrix} d_{L,1} \\ d_{L,2} \\ d_{L,3} \end{pmatrix} = V_{CKM} \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix}$$

"gauge eigenstates" "mass eigenstates"

While the gauge eigenstates have diagonal and universal couplings,

$$\mathcal{L} = \frac{g}{\sqrt{2}} W_\mu^+ \bar{u}_L^i \gamma^\mu d_{Li} + \frac{g}{\sqrt{2}} W_\mu^- \bar{d}_L^i \gamma^\mu u_{Li}$$

$$+ \frac{g}{c_W} Z_\mu \left[c_L^d \bar{d}_L^i \gamma^\mu d_{Li} + c_R^u \bar{d}_R^i \gamma^\mu d_{Ri} + \dots \right]$$

the mass eigenstates will have non-diagonal, non-universal interactions. Cabibbo's motivation was to explain why weak interactions are not universal in the quark sector & seem so different than in the leptonic sector.

- The lack of universality only appears in the charged current sector.

Since $V_{CKM}^{\dagger} V_{CKM} = 1$,

$$\bar{d}_L^i \gamma^\mu d_{Li} = (\bar{d}_L, \bar{s}_L, \bar{b}_L) \gamma^\mu \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix}$$

$$= \bar{d}_L \gamma^\mu d_L + \bar{s}_L \gamma^\mu s_L + \bar{b}_L \gamma^\mu b_L$$

Also Higgs couplings are diagonal.

Neutral gauge interactions are diagonal and universal.

Charged interactions, instead,

$$\mathcal{L} \supset \frac{g}{\sqrt{2}} (V_{CKM})_i{}^j W_\mu^+ \bar{u}_L^i \gamma^\mu d_{Lj}$$

$$+ \frac{g}{\sqrt{2}} (V_{CKM}^*)_j{}^i W_\mu^- \bar{d}_L^j \gamma^\mu u_{Li}$$

- Finally, some notation.

We denote the entries of the CKM matrix as

$$V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

in order to remind that each entry corresponds to a quark transition.

We can write V_{CKM} in a way that makes explicit the relations due to unitarity & removal of phases. A usual one is called Wolfenstein parametrization,

$$V_{CKM} = \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4)$$

with $\lambda \simeq \sin\theta_c \simeq 0.23$

$$A \simeq 0.8 \quad , \quad \rho \simeq 0.14 \quad , \quad \eta \simeq 0.35$$

- The parametrization shows that

  1-2 trans. >> 2-3 trans. >> 1-3 trans.

- Since $\cancel{CP}$ requires 3rd family, phase is associated with 1-3 entry.

- It is often a good approximation to work with

$$V_{CKM} \simeq \begin{pmatrix} V_{Cabibbo} & 0 \\ 0 & 1 \end{pmatrix}$$

i.e. to work with two families.